Ex. 62

1) Soit $P, Q \in \mathbb{R}[X]$
Soit $\lambda \in \mathbb{R}$

$$\forall x \in \mathbb{R},\ f(P+\lambda Q)(x) = \int_x^{x+1} (P+\lambda Q)(t)\,dt$$
$$= \int_x^{x+1} (P(t)+\lambda Q(t))\,dt$$
$$= \int_x^{x+1} P(t)\,dt + \lambda \int_x^{x+1} Q(t)\,dt$$
$$= f(P) + \lambda f(Q)$$

Soit $P \in \mathbb{R}[X]$ tq. $P = \sum_{k=0}^{n} a_k X^k$

$$\forall x \in \mathbb{R},\ f(P) = \int_x^{x+1} \sum_{k=0}^{n} a_k t^k\,dt$$
$$= \left[\sum_{k=0}^{n} \frac{a_k}{k+1} t^{k+1}\right]_x^{x+1}$$
$$= \sum_{k=0}^{n} \frac{a_k}{k+1}(x+1)^{k+1} - \sum_{k=0}^{n} \frac{a_k}{k+1} x^{k+1}$$
$$= \sum_{k=0}^{n} \frac{a_k}{k+1}\left((x+1)^{k+1} - x^{k+1}\right)$$
$$\in \mathbb{R}[X]$$

Donc $f : \mathbb{R}[X] \to \mathbb{R}[X]$

$$\boxed{\text{Ainsi, } f \in \mathcal{L}(\mathbb{R}[X])}$$

2) a) $\forall P \in \mathbb{R}_n[X],\ f_n(P) = f(P)$

Par 1), f_n est linéaire

$\forall P \in \mathbb{R}_n[X],\ f_n(P) = \sum_{k=0}^{n} \frac{a_k}{k+1}\left((x+1)^{k+1} - x^{k+1}\right)$

$\in \mathbb{R}_n[X]$ car les termes de degré $n+1$ se simplifient

$\boxed{\text{Ainsi, } f_n \in \mathcal{L}(\mathbb{R}_n[X])}$

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2) b) Soit $B = (1, X, X^2, \dots, X^n)$ base canonique de $\mathbb{R}_n[X]$

$\forall k \in \mathbb{N},\ f_n(X^k) = f(X^k)$

$$= \int_x^{x+1} t^k \, dt$$

$$= \left[\frac{t^{k+1}}{k+1}\right]_x^{x+1}$$

$$= \frac{(x+1)^{k+1} - x^{k+1}}{k+1}$$

$$= \frac{1}{k+1}\left(\sum_{i=0}^{k+1} \binom{k+1}{i} x^i - x^{k+1}\right)$$

$$= \frac{1}{k+1} \sum_{i=0}^{k} \binom{k+1}{i} x^i$$

On a $f_n(1) = 1$

$f_n(X) = \frac{1}{2} + x$

$f_n(X^2) = \frac{1}{3} + x + x^2$

etc...

D'où $\mathrm{Mat}_{\mathcal{B}}(f_u) = \begin{pmatrix} 1 & 1/2 & 1/3 & & & \\ 0 & 1 & 1 & & (*) & \\ \vdots & 0 & 1 & & & \\ & \vdots & 0 & \ddots & & \\ & & \vdots & (0) & & \\ 0 & 0 & 0 & & & 1 \end{pmatrix}$

$\boxed{\text{Ainsi, } \det(f_u) = 1}$

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2) c) Notons $A = \mathrm{Mat}_{\mathcal{B}}(f_u)$

On a $\chi_A = (X-1)^{n+1}$ ⚠ $\dim \mathbb{R}_n[X] = n+1$

D'où $Sp(A) = \{1\}$

A est diagonalisable ssi $A = I_n$

Or $A \neq I_n$ SAUF SI $n=0$

Ainsi, A ~~n'~~est ~~pas~~ diagonalisable ssi $n=0$.

FIN