17 À l'aide du changement de variable $(u,v)=\left(x,\dfrac{y}{x}\right)$, résoudre sur $\mathscr{C}^2(\mathscr{U})$ où $\mathscr{U}=\mathbb{R}_*^+\times\mathbb{R}$,

$$x^2\frac{\partial^2 f}{\partial x^2}(x,y)+2xy\frac{\partial^2 f}{\partial x\partial y}(x,y)+y^2\frac{\partial^2 f}{\partial y^2}(x,y)=0$$

$$\varphi : \left|\begin{array}{l} \mathbb{R}_+^*\times\mathbb{R} \longrightarrow \mathbb{R}_+^*\times\mathbb{R} \\ (x,y) \longmapsto \left(x,\frac{y}{x}\right) = (u,v)\end{array}\right. \qquad \varphi, \varphi^{-1} \quad \mathscr{C}^2.$$

bijective $\varphi^{-1} : \left|\begin{array}{l} \mathbb{R}_+^*\times\mathbb{R} \longrightarrow \mathbb{R}_+^*\times\mathbb{R} \\ (u,v) \longmapsto (u, u\times v)\end{array}\right.$

$$f(x,y) = g(u,v) = g\left(x,\frac{y}{x}\right)$$

f de classe $\mathscr{C}^2$ sur $\mathbb{R}_+^*\times\mathbb{R}$ ssi g l'est

$$\frac{\partial f}{\partial x}(x,y) = 1\times\frac{\partial g}{\partial u}\left(x,\frac{y}{x}\right) - \frac{y}{x^2}\frac{\partial g}{\partial v}\left(x,\frac{y}{x}\right)$$

$$\frac{\partial f}{\partial y}(x,y) = \frac{1}{x}\frac{\partial g}{\partial v}\left(x,\frac{y}{x}\right)$$

$$x^2\times \quad \frac{\partial^2 f}{\partial x^2}(x,y) = \frac{\partial^2 g}{\partial u^2}\left(x,\frac{y}{x}\right) - \frac{y}{x^2}\frac{\partial^2 g}{\partial v\partial u}\left(x,\frac{y}{x}\right) + \frac{2y}{x^3}\frac{\partial g}{\partial v}\left(x,\frac{y}{x}\right) - \frac{y}{x^2}\left[\frac{\partial^2 g}{\partial u\partial v}\left(x,\frac{y}{x}\right) - \frac{y}{x^2}\frac{\partial^2 g}{\partial v^2}\left(x,\frac{y}{x}\right)\right]$$

$$2xy\times \quad \frac{\partial^2 f}{\partial x\partial y}(x,y) = -\frac{1}{x^2}\frac{\partial g}{\partial v}\left(x,\frac{y}{x}\right) + \frac{1}{x}\left[\frac{\partial^2 g}{\partial u\partial v}\left(x,\frac{y}{x}\right) - \frac{y}{x^2}\frac{\partial^2 g}{\partial v^2}\left(x,\frac{y}{x}\right)\right]$$

$$y^2\times \quad \frac{\partial^2 f}{\partial y^2}(x,y) = \frac{1}{x^2}\frac{\partial^2 g}{\partial v^2}\left(x,\frac{y}{x}\right)$$

f solu° sur $\mathbb{R}_+^* \times \mathbb{R} \iff \forall (x,y) \in \mathbb{R}_+^* \times \mathbb{R},\ x^2 \frac{\partial^2 g}{\partial u^2}\left(x, \frac{y}{x}\right) + \cancel{\left[-2y + 2y\right] \frac{\partial^2 g}{\partial u \partial v}\left(x, \frac{y}{x}\right)} + \cancel{\left[\frac{y^2}{x^2} - \frac{2y^2}{x^2} + \frac{y^2}{x^2}\right]} \frac{\partial^2 g}{\partial v^2}\left(x, \frac{y}{x}\right)$

$$+ \cancel{\left[\frac{2y}{x} - \frac{2y}{x}\right] \frac{\partial g}{\partial v}\left(x, \frac{y}{x}\right)} = 0$$

$$\iff \forall (u,v) \in \mathbb{R}_+^* \times \mathbb{R},\ \frac{\partial^2 g}{\partial u^2}(u,v) = 0$$

$$\iff \exists \varphi, \psi \in \mathcal{C}^2(\mathbb{R}),\ \forall (u,v) \in \mathbb{R}_+^* \times \mathbb{R},\ g(u,v) = \varphi(v)\,u + \psi(v)$$

car $\mathbb{R}_+^*$ intervalle

$$\iff \boxed{\exists \varphi, \psi \in \mathcal{C}^2(\mathbb{R}),\ \forall (x,y) \in \mathbb{R}_+^* \times \mathbb{R},\ f(x,y) = \varphi\left(\frac{y}{x}\right) x + \psi\left(\frac{y}{x}\right).}$$

28 **Fonctions harmoniques** Une application $f : \mathscr{U} \mapsto \mathbb{R}$, de classe $\mathscr{C}^2$ sur un ouvert $\mathscr{U}$ de $\mathbb{R}^2$ est dite ***harmonique*** si et seulement si $\Delta f = 0$ où $\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$ est le ***laplacien*** de f.

1. Pour $(x, y) \in \mathbb{R}^2$, soient $z = x + iy \in \mathbb{C}$ et $f(x, y) = \ln\left|e^{ze^{-z}}\right|$. Montrer que f est harmonique sur $\mathbb{R}^2$.
2. Montrer que si f est de classe $\mathscr{C}^3$ et harmonique, alors $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, y\frac{\partial f}{\partial x} - x\frac{\partial f}{\partial y}$ sont harmoniques.
3. Vérifier que $f : (x, y) \mapsto \operatorname{Arctan} \frac{y}{x}$ est harmonique sur $\mathbb{R}^* \times \mathbb{R}$.

$|e^{a+ib}| = e^a > 0$

$|e^{ib}| = 1$

1 - $f(x,y) = \ln\left(e^{\mathrm{Re}(ze^{-z})}\right) = \mathrm{Re}(ze^{-z}) = \mathrm{Re}\left((x+iy)\,e^{-x-iy}\right)$

$$= e^{-x}\,\mathrm{Re}\Big((x+iy)(\cos y - i \sin y)\Big)$$

$$= e^{-x}\left[x \cos y + y \sin y\right]$$

f de classe $\mathcal{C}^2$ sur $\mathbb{R}^2$.

$$\frac{\partial f}{\partial x} : (x,y) \longmapsto e^{-x}\left[-x\cos y - y\sin y + \cos y\right]$$

$$\frac{\partial f}{\partial y} (x,y) \longmapsto e^{-x}\left[-x\sin y + \sin y + y\cos y\right]$$

$$\frac{\partial^2 f}{\partial x^2} : (x,y) \longmapsto e^{-x}\left[-(-x\cos y - y\sin y + \cos y) - \cos y\right]$$

$$\frac{\partial^2 f}{\partial y^2} : (x,y) \longmapsto e^{-x}\left[-x\cos y + \cos y + \cos y - y\sin y\right]$$

donc $\Delta f = 0$.

2 - $\Delta f = 0$ $\quad f \in \mathcal{C}^3(\mathbb{R}^2)$

$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ $\mathcal{C}^2$

Schwarz, f $\mathcal{C}^3$

$$\Delta\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2}{\partial x^2}\left[\frac{\partial f}{\partial x}\right] + \frac{\partial^2}{\partial y^2}\left[\frac{\partial f}{\partial x}\right] = \frac{\partial}{\partial x}\left[\frac{\partial^2 f}{\partial x^2}\right] + \frac{\partial}{\partial x}\left[\frac{\partial^2 f}{\partial y^2}\right] = \frac{\partial}{\partial x}\left[\underbrace{\Delta(f)}_{=0}\right] = 0.$$

$\Delta\left(\frac{\partial f}{\partial y}\right) = 0$ par symétrie.

$$g : (x,y) \longmapsto y\frac{\partial f}{\partial x}(x,y) - x\frac{\partial f}{\partial y}(x,y) \quad \mathcal{C}^2$$

$$\frac{\partial g}{\partial x} : (x,y) \longmapsto y\frac{\partial^2 f}{\partial x^2}(x,y) - \frac{\partial f}{\partial y}(x,y) - x\frac{\partial^2 f}{\partial x \partial y}(x,y)$$

$$\frac{\partial g}{\partial y} : (x,y) \longmapsto \frac{\partial f}{\partial x}(x,y) + y\frac{\partial^2 f}{\partial x\partial y}(x,y) - x\frac{\partial^2 f}{\partial y^2}(x,y)$$

$\forall (x,y) \in \mathbb{R}^2$,

$$\Delta g(x,y) = \left[y\frac{\partial^3 f}{\partial x^3}(x,y) \cancel{-2\frac{\partial^2 f}{\partial x\partial y}(x,y)} - x\frac{\partial^3 f}{\partial x^2\partial y}(x,y)\right] + \left[\cancel{2\frac{\partial^2 f}{\partial x\partial y}(x,y)} + y\frac{\partial^3 f}{\partial x\partial y^2}(x,y) - x\frac{\partial^3 f}{\partial y^3}(x,y)\right]$$

$$= y\frac{\partial}{\partial x}\Delta(f)(x,y) - x\frac{\partial}{\partial y}\Delta(f)(x,y)$$

$$= 0.$$

38

1 – $f : (x,y) \longmapsto x^2 + xy + y^2 - 3x - 6y$ de classe $\mathcal{C}^2$ sur $\mathbb{R}^2$ <u>ouvert</u>.

Si f présente un ext. local en (x,y) alors $\nabla f(x,y) = (0,0)$

$$\begin{cases} \dfrac{\partial f}{\partial x}(x,y) = 2x + y - 3 \\ \dfrac{\partial f}{\partial y}(x,y) = x + 2y - 6 \end{cases}$$

$$\frac{\partial^2 f}{\partial x^2}(x,y) = 2$$

$$\frac{\partial^2 f}{\partial x \partial y}(x,y) = 1$$

$$\frac{\partial^2 f}{\partial y^2}(x,y) = 2$$

$$\nabla f(x,y) = (0,0) \iff \begin{cases} 2x + y = 3 \\ x + 2y = 6 \end{cases} \iff (x,y) = (0,3)$$

$$H_f(0,3) = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

de det $3 > 0$ donc f présente un ext. local en $(0,3)$.

de tr $4 > 0$ donc $\mathrm{Sp}\, H_f(0,3) \subset \mathbb{R}_+^*$: min local.